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Skolem Function Continuation for Quantified Boolean Formulas

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JOHANNES KEPLER UNIVERSITY LINZ

LOGICAL METHODS IN COMPUTER SCIENCE

Austrian Rigorous Systems Engineering

INTRODUCTION - QBF

Quantified Boolean Formulas (QBF):

- Extension of propositional logic
  - Boolean variables
  - Logical connectives
  - Quantifiers ($\forall$, $\exists$) over the Boolean variables
- Harder to decide satisfiability (PSPACE-complete)
- Shorter encoding than SAT (NP-complete)

QBFs in Formal Verification

- Bounded Model Checking:
  - Aim: discover undesired behaviours of systems
  - Given a model for a system and a set of bad states
  - Starting from an initial state, is there a bad state that is reachable in k (or less) steps?

JKU

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- Synthesis

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- Synthesis
- Equivalence checking
- ...

QBF Syntax

$$\overbrace{\exists x \exists y \forall u \exists z.(u \rightarrow z) \wedge (y \vee u \vee \neg z) \wedge (x \vee \neg u \vee \neg z) \wedge (x \leftrightarrow \neg y)}^{\text{closed QBF in prenex form}}$$

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- **QBFs in Prenex CNF (PCNF):**

$$\exists x \exists y \forall u \exists z.(\neg u \vee z) \wedge (y \vee u \vee \neg z) \wedge (x \vee \neg u \vee \neg z)$$

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$$x = \mathsf{T}$$

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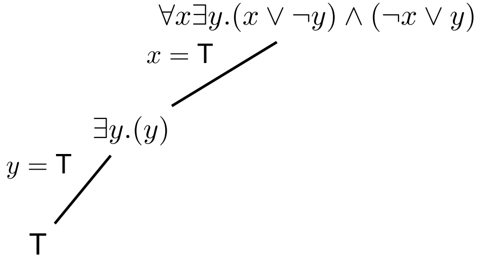

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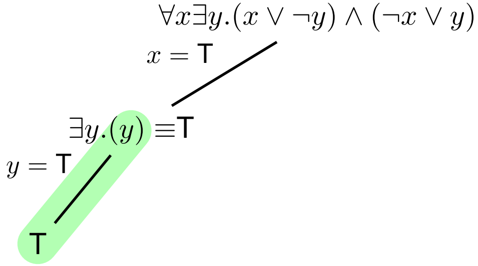

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$x = \mathsf{T}$: $\exists y.(y) \equiv \mathsf{T}$

$y = \mathsf{T}$: T

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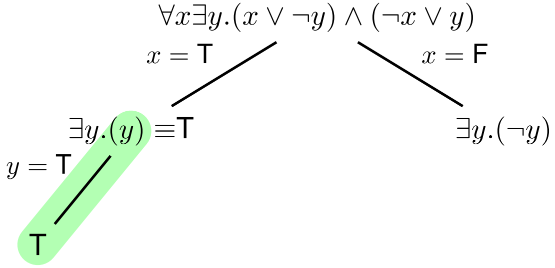

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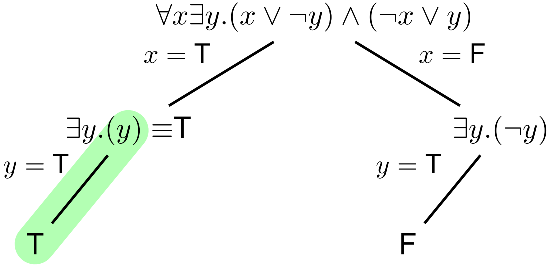

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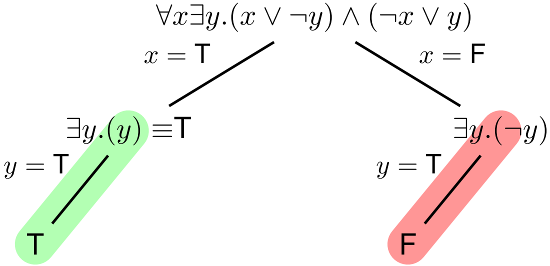

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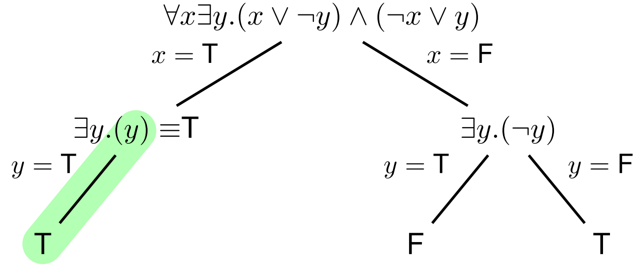

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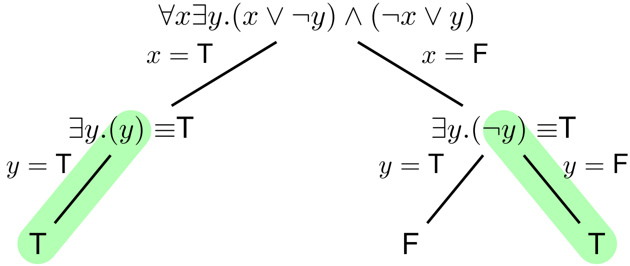

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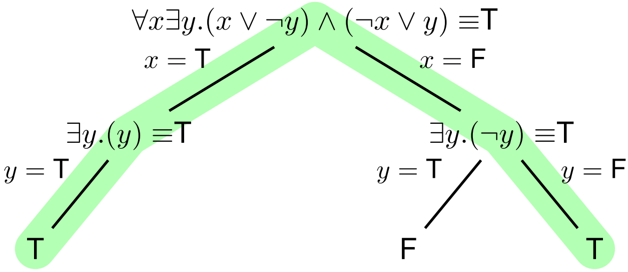

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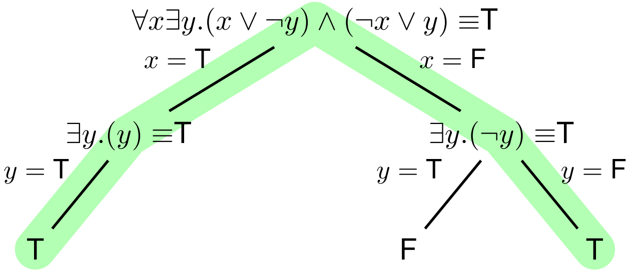

- Skolem-functions of $\exists$-variables:

$$sk_y(x)$$

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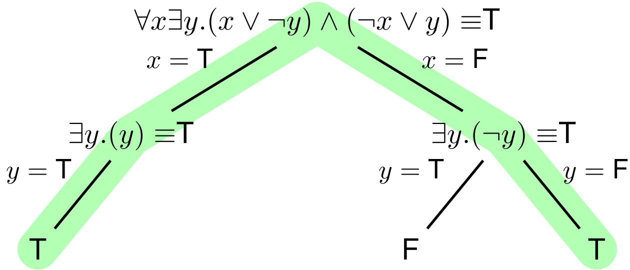

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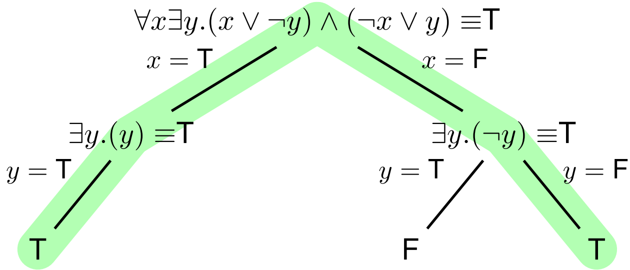

- Skolem-functions of $\exists$-variables:

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Skolem Functions

- Function
  - For each existential variable
  - Input arguments: in the prefix preceding $\forall$-variables
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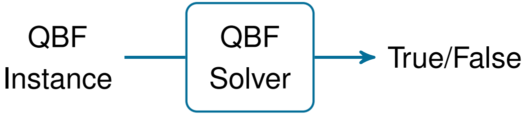
- Skolem functions as solution:
  - Bounded model checking: erroneous path

QBF PREPROCESSING & SOLVING

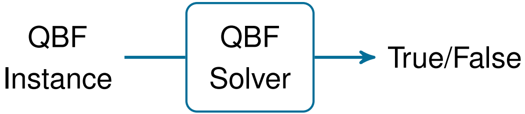
QBF Solvers

- Evaluate QBFs



QBF Solvers

- Evaluate QBFs
- Several tools: depQBF, CAQE, QuBE, sKizzo, RAReQS, ...



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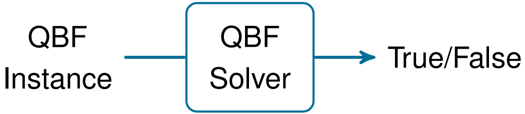

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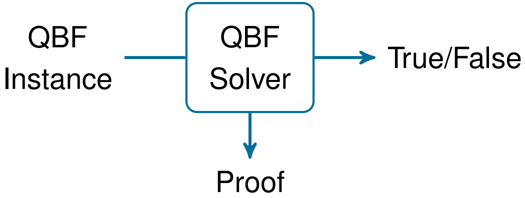

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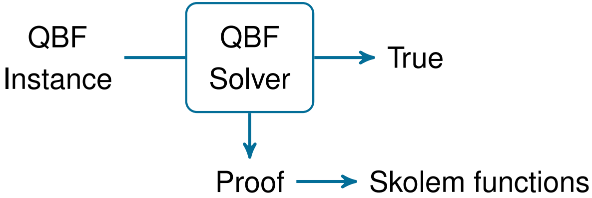

QBF Solvers

- Evaluate QBFs
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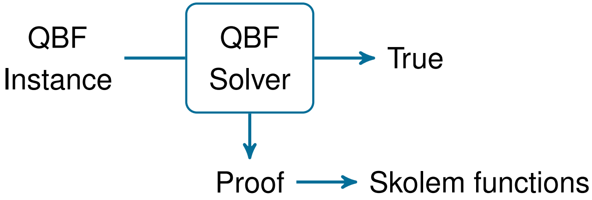
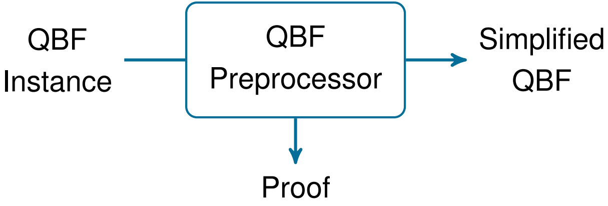
- Some problem instances are challenging



QBF Preprocessors

- Simplify QBFs



QBF Preprocessors

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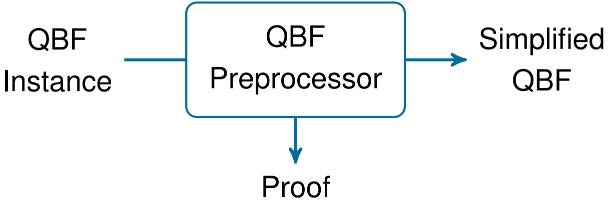

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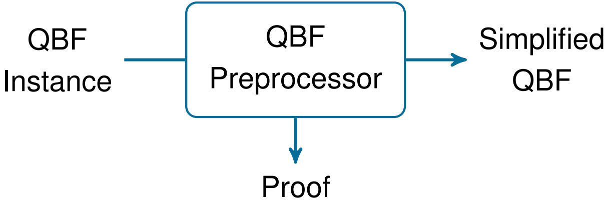

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- When QBF is simplified to True:
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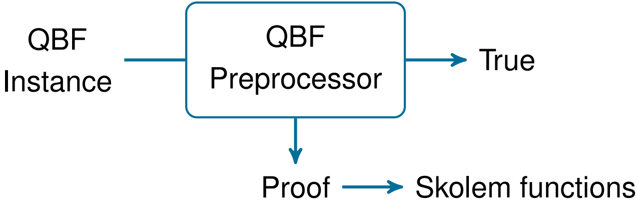

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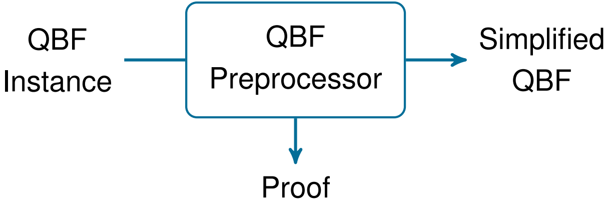
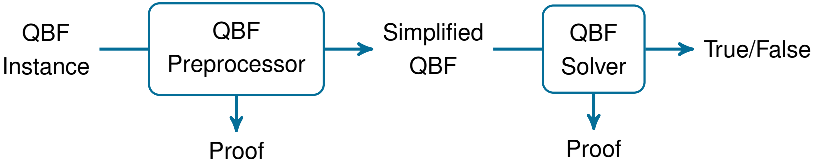
- Not model-preserving simplification steps



QBF Solving with Preprocessors and Solvers

- What if preprocessor simplified but did not solve the QBF?



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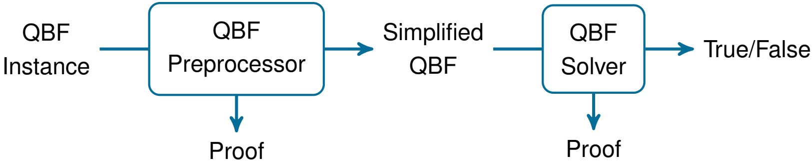

QBF Solving with Preprocessors and Solvers

- What if preprocessor simplified but did not solve the QBF?
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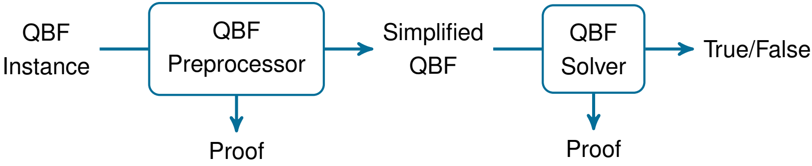
- Solution: Skolem function continuation



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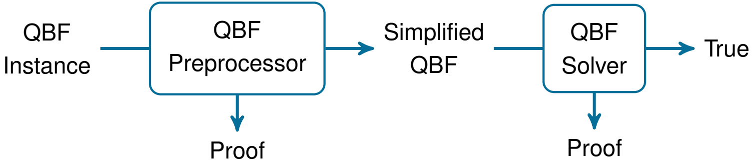

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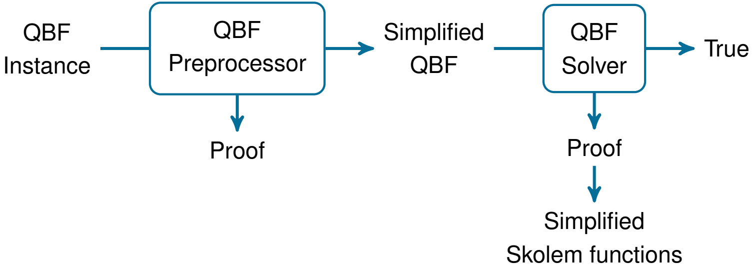

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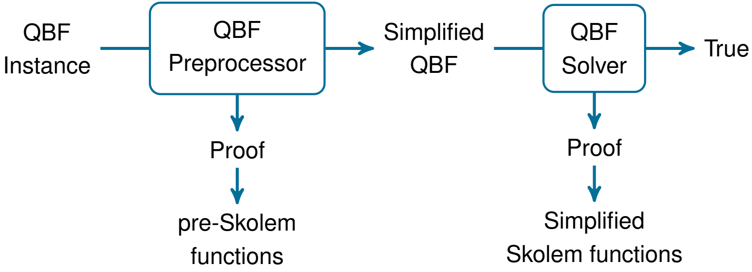

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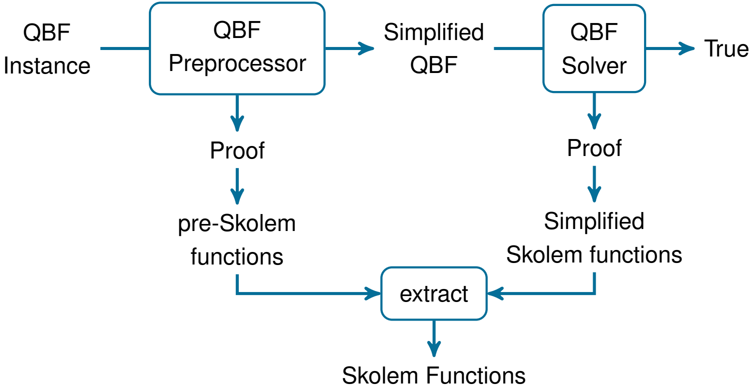

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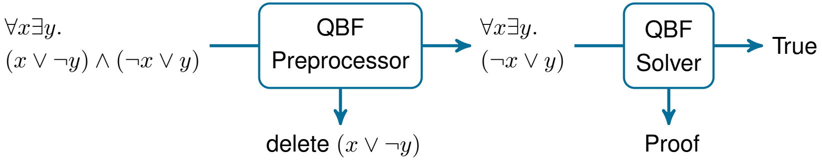

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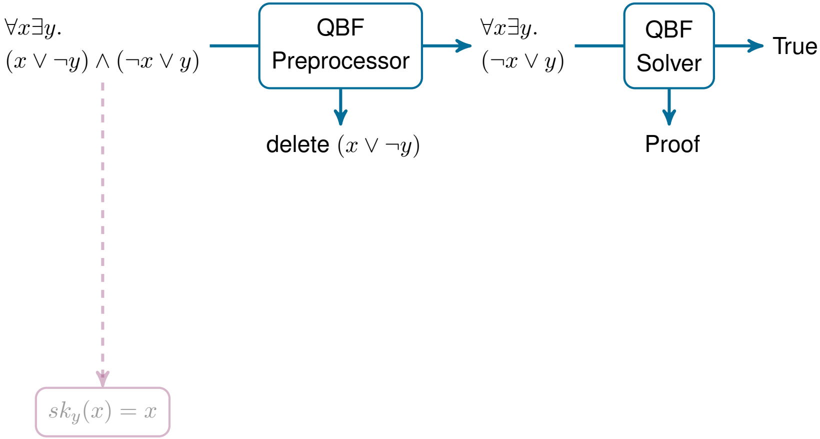

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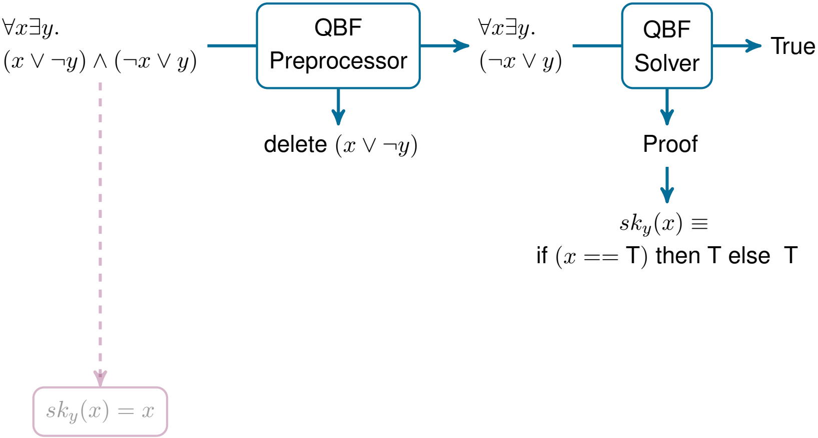

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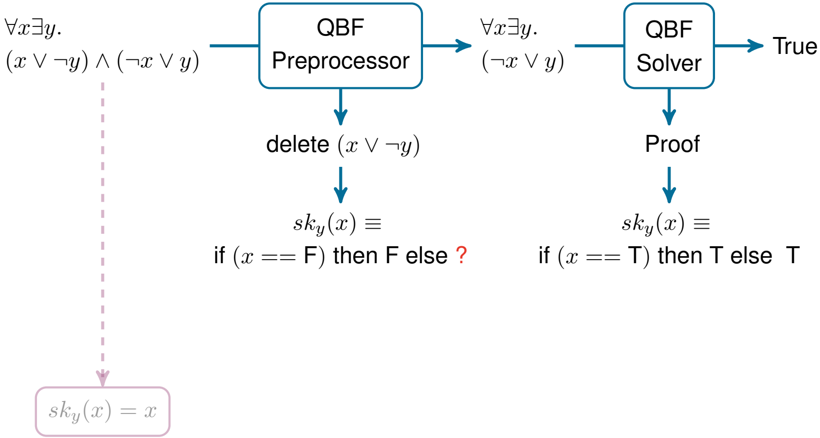

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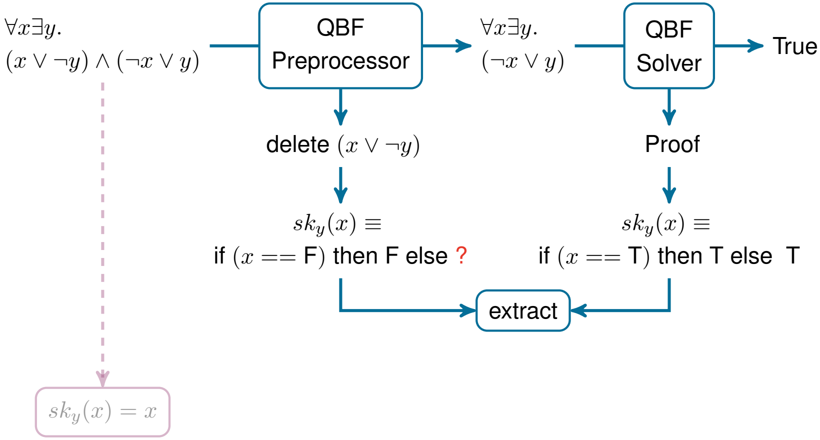
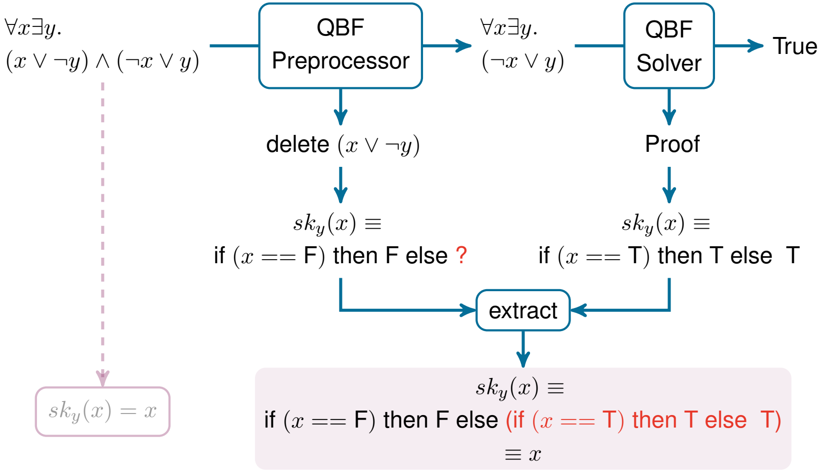


Skolem Function Continuation



Implementation

- New tool: **sk-extract**[1]
- Smooth integration into typical QBF solving tool chains
- Performs similarly well as the only available specialized approach
- Evaluation: QBF Eval 2016 main track (competition of QBF solvers)

[1] http://fmv.jku.at/sk-extract/

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- Future work:
  - Skolem function optimization