JOHANNES KEPLER
UNIVERSITY LINZ

A Duality-Aware Calculus for Quantified Boolean Formulas

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24. September, 2016

18th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing

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logics LOGICAL METHODS IN COMPUTER SCIENCE

ARiSE Austrian Rigorous Systems Engineering

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INTRODUCTION

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- Extension of propositional logic with explicit quantifiers ($\forall$, $\exists$) over the variables
- Canonical PSPACE-complete problem: more succinct encoding than SAT
- Several application domains: synthesis, AI, verification, ...

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JKU

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JKU

Introduction (2)

- QBFs in Prenex CNF (PCNF):

$$\exists x \exists y \forall u \exists z . (\neg u \vee z) \wedge (y \vee u \vee \neg z) \wedge (x \vee \neg u \vee \neg z)$$

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- QBFs in Prenex DNF (PDNF):

$$\forall x \forall y \exists u \forall z.(u \wedge \neg z) \vee (\neg y \wedge \neg u \wedge z) \vee (\neg x \wedge u \wedge z)$$

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$$\forall x \forall y \exists u \forall z.\underbrace{\overbrace{(u \wedge \neg z)}^{\text{cube}} \vee (\neg y \wedge \neg u \wedge z) \vee (\neg x \wedge u \wedge z)}_{\text{DNF}}$$

Introduction (3)

PCNF (PDNF) formula under assignment

- remove clauses (cubes) with satisfied (falsified) literals
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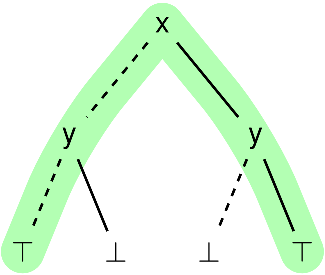
Semantics of QBF

- QBF $\forall x \mathcal{Q}.\varphi$ is satisfiable iff $\mathcal{Q}.\varphi[x]$ **and** $\mathcal{Q}.\varphi[\neg x]$ are satisfiable
- QBF $\exists x \mathcal{Q}.\varphi$ is satisfiable iff $\mathcal{Q}.\varphi[x]$ **or** $\mathcal{Q}.\varphi[\neg x]$ is satisfiable

Introduction (4)

Tree model of true formula:

$$\forall x \exists y.(x \vee \bar{y}) \wedge (\bar{x} \vee y)$$



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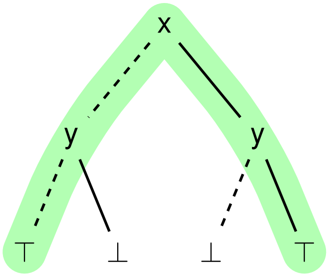
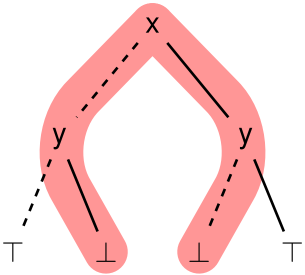


Tree refutation of false formula:

$\exists x \forall y.(x \vee \bar{y}) \wedge (\bar{x} \vee y)$



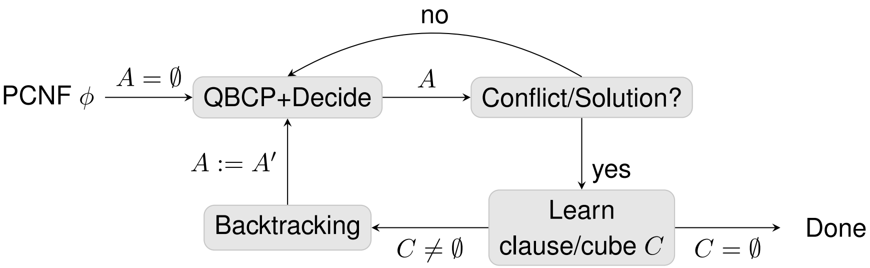
SEARCH-BASED QBF SOLVING

Search-Based QBF Solving: QCDCL (1)

```
Result qcdcl (PCNF φ)
  Result R = UNDEF;
  Assignment A = ∅;
  while (true)
    /* Simplify under A. */
    (R,A) = qbcp(φ,A);
    if (R == UNDET)
      /* Decision making. */
      A = assign_dec_var(φ,A);
    else
      /* Backtracking. */
      /* R == UNSAT/SAT */
      B = analyze(R,A);
      if (B == INVALID)
        return R;
      else
        A = backtrack(B);
```

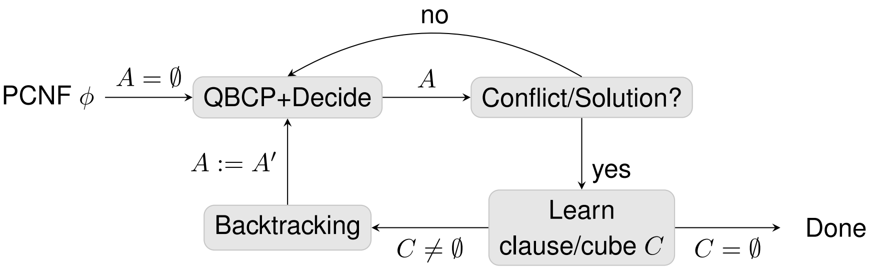
- QBF-specific version of SAT CDCL algorithm
- expects the problem to be formulated in PCNF
- traverses the assignment tree of the input formula
- conflict analysis similar to SAT solvers
- satisfaction recognition requires additional efforts

Search-Based QBF Solving: QCDCL (2)



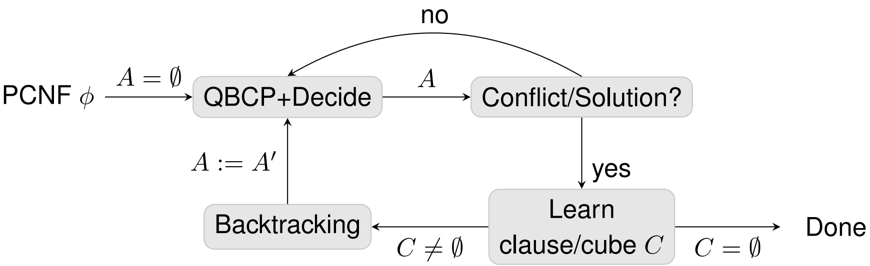
1. Construct assignment (A) by QBF specific propagation and decisions.

Search-Based QBF Solving: QCDCL (2)



1. Construct assignment (A) by QBF specific propagation and decisions.
2. Check the followings:
   - ☐ Is there a falsified clause under A and universal reduction? (Conflict)
   - ☐ Are all the clauses of the formula satisfied under A? (Solution)

Search-Based QBF Solving: QCDCL (2)



3. Derive a clause (cube) C from A and ϕ and learn it.

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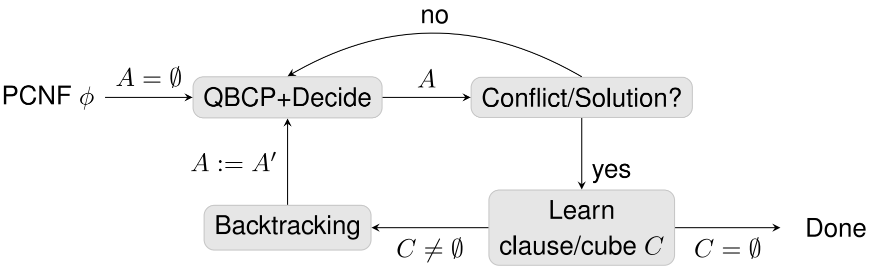

3. Derive a clause (cube) C from A and ϕ and learn it.
4. Use the learned clause (cube) to backtrack.
   - ☐ $C = \emptyset$: no place to backtrack, the formula is UNSAT (SAT).
   - ☐ $C \neq \emptyset$: C is 'driving' the backtracking.

Observations

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- How to analyse the behaviour of search-based QBF solvers?
- How to compare them with other QBF solving approaches?
- How to verify them?

ABSTRACT QBF SOLVING

Duality-Aware Abstract QBF Solver

Abstract Solver:

- Describes the essential properties of QBF solvers similar to the well-known DPLL transition system
- Provides a framework for analysing, comparing and composing solvers without technical details
- Duality-Aware: Conflicts and solutions are handled in the same way

State Transition System as QBF Solver

Possible states of a QBF solver: $A \parallel \mathcal{D} \parallel \mathcal{C}$

- A quantifier prefix: $\mathcal{Q}$
- The current assignment (sequence of literals): A
- A set of cubes: $\mathcal{D}$
- A set of clauses: $\mathcal{C}$

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Possible steps of a QBF solver:

- Transition relation over the states defined by conditional transition rules
- Different specializations, heuristics can be seen as refinements of the relations

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- Transition relation over the states defined by conditional transition rules
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Describe the solving process as a derivation in the calculus.

Calculus Rules

$$\frac{A \parallel \mathcal{D} \parallel \mathcal{C} \wedge C}{A\ \ell_\exists \parallel \mathcal{D} \parallel \mathcal{C} \wedge C}\ \ell_\exists \text{ existential unit in } C[A] \qquad (\mathrm{Unit}_\exists)$$

$$\frac{A \parallel \mathcal{D} \vee C \parallel \mathcal{C}}{A\ \neg\ell_\forall \parallel \mathcal{D} \vee C \parallel \mathcal{C}}\ \ell_\forall \text{ universal unit in } C[A] \qquad (\mathrm{Unit}_\forall)$$

$$\frac{A \parallel \mathcal{D} \parallel \mathcal{C}}{A\ \ell_\exists \parallel \mathcal{D} \parallel \mathcal{C}}\ \ell_\exists \in \mathcal{R}_\exists^\mathcal{Q}(\mathcal{D}[A]) \text{ is pure} \qquad (\mathrm{Pure}_\exists)$$

$$\frac{A \parallel \mathcal{D} \parallel \mathcal{C}}{A\ \neg\ell_\forall \parallel \mathcal{D} \parallel \mathcal{C}}\ \ell_\forall \in \mathcal{R}_\forall^\mathcal{Q}(\mathcal{C}[A]) \text{ is pure} \qquad (\mathrm{Pure}_\forall)$$

$$\frac{A \parallel \mathcal{D} \parallel \mathcal{C}}{A \parallel \mathcal{D} \parallel \mathcal{C} \wedge C}\ \mathcal{C} \vDash_\mathcal{Q} C \qquad (\mathrm{Learn}_{\mathrm{CNF}})$$

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$$\frac{A\ \ell_\exists^d A' \parallel \mathcal{D} \parallel \mathcal{C}}{A \parallel \mathcal{D} \parallel \mathcal{C}} \qquad (\mathrm{Undo}_\exists)$$

$$\frac{A\ \ell_\forall^d A' \parallel \mathcal{D} \parallel \mathcal{C}}{A \parallel \mathcal{D} \parallel \mathcal{C}} \qquad (\mathrm{Undo}_\forall)$$

$$\frac{A \parallel D \parallel \mathcal{C} \wedge \emptyset}{\bot} \qquad (\mathrm{Final}_{\mathrm{CNF}})$$

$$\frac{A \parallel \mathcal{D} \vee \emptyset \parallel \mathcal{C}}{\top} \qquad (\mathrm{Final}_{\mathrm{DNF}})$$

$$\frac{A \parallel \mathcal{D} \parallel \mathcal{C}}{A\ \ell^d \parallel \mathcal{D} \parallel \mathcal{C}}\ \ell \text{ is unassigned and all } \ell' \text{ with } \ell' <_\mathcal{Q} \ell \text{ are assigned in } A \qquad (\mathrm{Decide})$$

Remarks

- Strategy: Additional constraints in order to guarantee termination and to make the solver more realistic.

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- Extendable: further rules to represent functionalities of practical solvers (e.g. forget, restart).
- If duality can not be assumed, it is possible to easily adopt the system for PCNF-based solvers.

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$$\exists x \forall y . x \Leftrightarrow y$$

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- Starting state:
  $S = \emptyset \parallel \mathcal{D} \parallel \mathcal{C}$

Example (2)

$\mathcal{Q} = \exists x \forall y \exists p \forall q$

A	$\mathcal{D}$	$\mathcal{C}$
	q	p
$\emptyset$	$\neg x \wedge \neg y \wedge \neg q$	$\neg x \vee y \vee \neg p$
	$x \wedge y \wedge \neg q$	$x \vee \neg y \vee \neg p$

$\emptyset \parallel \mathcal{D} \parallel \mathcal{C}$

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Unit$_\exists$:

$$\frac{A \parallel \mathcal{D} \parallel \mathcal{C} \wedge C}{A\ \ell_\exists \parallel \mathcal{D} \parallel \mathcal{C} \wedge C}$$

$\ell_\exists$ existential unit in $C[A]$

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A	$\mathcal{R}^{\mathcal{Q}}_{\exists}(\mathcal{D}[A])$	$\mathcal{R}^{\mathcal{Q}}_{\forall}(\mathcal{C}[A])$
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Learn$_{\text{CNF}}$:

$$\frac{A \parallel \mathcal{D} \parallel \mathcal{C}}{A \parallel \mathcal{D} \parallel \mathcal{C} \wedge C}$$

$$\mathcal{C} \models_\mathcal{Q} C$$

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$$\mathcal{C} \vDash_{\mathcal{Q}} C$$

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	$x \wedge y \wedge \neg q$	$x \vee \neg y \vee \neg p$
		$\emptyset$

$$\cfrac{\cfrac{\cfrac{\cfrac{\emptyset \parallel \mathcal{D} \parallel \mathcal{C}}{p \parallel \mathcal{D} \parallel \mathcal{C}}\ \text{Unit}_{\exists}}{p\ \neg q \parallel \mathcal{D} \parallel \mathcal{C}}\ \text{Unit}_{\forall}}{p\ \neg q\ x \parallel \mathcal{D} \parallel \mathcal{C}}\ \text{Unit}_{\exists}}{p\ \neg q\ x \parallel \mathcal{D} \parallel \mathcal{C} \wedge \emptyset}\ \text{Learn}_{\text{CNF}}$$

$\text{Final}_{\text{CNF}}$:

$$\frac{A \parallel D \parallel \mathcal{C} \wedge \emptyset}{\bot}$$

Example (2)

$\mathcal{Q} = \exists x \forall y \exists p \forall q$

A	$\mathcal{R}^{\mathcal{Q}}_{\exists}(\mathcal{D}[A])$	$\mathcal{R}^{\mathcal{Q}}_{\forall}(\mathcal{C}[A])$
	q	
$p \ \neg q \ x$	$\neg x \wedge \neg y \wedge \neg q$	$\bot$
	$x \wedge y \wedge \neg q$	

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\emptyset \parallel \mathcal{D} \parallel \mathcal{C}}{p \parallel \mathcal{D} \parallel \mathcal{C}}\ \text{Unit}_{\exists}}{p\ \neg q \parallel \mathcal{D} \parallel \mathcal{C}}\ \text{Unit}_{\forall}}{p\ \neg q\ x \parallel \mathcal{D} \parallel \mathcal{C}}\ \text{Unit}_{\exists}}{p\ \neg q\ x \parallel \mathcal{D} \parallel \mathcal{C} \wedge \emptyset}\ \text{Learn}_{\text{CNF}}}{\bot}\ \text{Final}_{\text{CNF}}$$

$\text{Final}_{\text{CNF}}$:

$$\frac{A \parallel D \parallel \mathcal{C} \wedge \emptyset}{\bot}$$

Conclusion

Abstract search-based QBF solvers

- Simple formalism to describe the behavior of search-based QBF solvers without the technical details
- Provides better understanding of individual solving techniques
- Flexible representation: specialization of calculus rules to describe e.g. different decision heuristics
- Step towards verified QBF solvers

Future work

- Formalize preprocessing techniques
- Comparison to non-QCDCL solvers